Non-Intrusive and Intrusive Stochastic Finite Integration Technique for Magnetostatics

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Both non-intrusive and intrusive stochastic approaches based on Polynomial Chaos Expansions are presented for the Finite Integration Technique over generic polyhedral grids for three-dimensional magnetostatic linear problems. Such algorithms outperform Monte Carlo methods, both in terms of accuracy and efficiency. A novel efficient algorithm for the intrusive approach is also provided, by which the intrusive approach becomes much less computationaly expensive than the non-intrusive approach. Validation is carried out by solving a magnetic circuit where the reluctivity is uncertain.

***Index Terms*—Magnetostatics, Finite Integration Technique, Uncertainty.**

I. Introduction

THIS PAPER introduces and compares non-intrusive and intrusive stochastic approaches for the Finite Integration Technique (FIT) based on Polynomial Chaos Expansion (PCE) and applied to magnetostatics problems in which magnetic reluctivity is random.

Corresponding formulations for electrokinetic and eddy current problems were introduced by the authors in [1], [2]. All these formulations bring to the stochastic domain the main properties of deterministic FIT discretizations, such as that of combining exact balance equations with approximate constitutive equations, over pairs of arbitrary polyhedral dual grids. They also bring all benefits of PCE with respect to Monte Carlo (MC) methods, in terms both of accuracy and efficiency.

In this paper a novel effective algorithm is also provided for the intrusive approach, allowing to efficiently store and solve the attained linear system of equations in such a way that the intrusive approach becomes the premier stochastic FIT approach for the chosen application. As a numerical validation, a typical geometry of a magnetic circuit (Fig. 1) is considered. In this problem the reluctivities of the four blocks composing the magnetic circuit are modeled as uniformly distributed random variables [3]. A total DC current is imposed on surface σ of a rectangular coil Ω_C and the aim is to statistically characterize the magnetic flux through section Σ. Regions Ω_k, with $k = 1, \ldots, 4$, are assumed to have statistically independent reluctivities with uniformly distributed probability density functions in the ranges reported in the caption of Fig. 1.

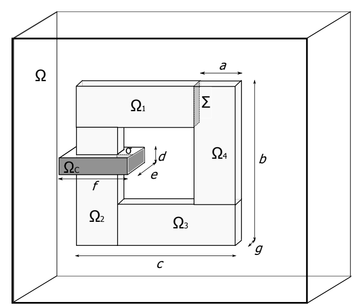

Fig. 1. Geometry of the problem: a=50 mm, b=c=200 mm, d=10 mm, e=30 mm, f=70 mm, g=10 mm, $\nu_1 = 0.1 \pm 0.01$mm/H, $\nu_2 = 0.05 \pm 0.005$mm/H, $\nu_3 = 0.033 \pm 0.0033$mm/H, $\nu_4 = 0.025 \pm 0.0025$mm/H.

II. Deterministic FIT formulation

Deterministic magnetostatic problems can be discretized by FIT in various ways. In a vector potential formulation, hereinafter considered, a pair of arbitrary polynomial dual grids $\mathcal{G}$, $\tilde{\mathcal{G}}$ is constructed. Array $\boldsymbol{b}$, with the fluxes of magnetic induction through the faces, and array $\boldsymbol{a}$ with the circulations of the vector potential along the edges, are defined over the primal grid $\mathcal{G}$. Similarly array $\tilde{\boldsymbol{h}}$, with the circulations of the magnetic field along the edges, and array $\tilde{\boldsymbol{j}}$, with the fluxes of current density through the faces, are defined over dual grid $\tilde{\mathcal{G}}$. Discretizing Ampére's law and the solenoidality of magnetic induction, discretizing magnetic constitutive equations by the energetic approach [4], and combining equations it ensues

$$\mathbf{C}^T \mathbf{M}_\nu \mathbf{C} \boldsymbol{a} = \tilde{\boldsymbol{j}}, \tag{1}$$

in which $\mathbf{C}$ is the face-edge incidence matrix of $\mathcal{G}$ and $\mathbf{M}_\nu$ is the symmetric, positive definite discrete reluctivity matrix.

Even if underdetermined, this equation can be efficiently solved by iterative methods, in particolar by the conjugate gradient method.

III. Stochastic FIT formulation

Let the magnetic reluctivity ν be dependent on a small number q of random variables, assumed to be statistically independent and forming a vector $\boldsymbol{\xi}$. Thus it is written $\nu = \nu(\mathbf{r}, \boldsymbol{\xi})$, in which $\mathbf{r}$ is the position vector.

PCEs can be introduced for all discrete variables of FIT. In this way, for instance, array $\boldsymbol{a}(\boldsymbol{\xi})$ of the circulations of the vector potential is approximated in the form

$$\boldsymbol{a}(\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leq p} \boldsymbol{a}_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{2}$$

in which $\boldsymbol{\alpha}$ is a multi-index of q elements, and $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ are polynomials of degrees less than p, forming an orthonormal basis in the probability space of the random vector $\boldsymbol{\xi}$ [3], [5].

All the n projections $\boldsymbol{a}_{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}|\leq p$ can be grouped in vector $\boldsymbol{A} = [\boldsymbol{a}_{\boldsymbol{\alpha}}]$. In a similar way all the other FIT variables, assumed random, can be represented by their PCEs $\tilde{\boldsymbol{J}} = [\tilde{\boldsymbol{j}}_{\boldsymbol{\alpha}}]$, $\tilde{\boldsymbol{H}} = [\tilde{\boldsymbol{h}}_{\boldsymbol{\alpha}}]$, $\boldsymbol{B} = [\boldsymbol{b}_{\boldsymbol{\alpha}}]$.

A *non-intrusive* stochastic approach for FIT can be obtained by solving deterministic FIT magnetostatic problem (1) for a proper set of choices of random variables and by reconstructing the projections of the FIT discrete variables, by either interpolation, collocation or pseudo-spectral method [5].

Conversely, an *intrusive* stochastic approach is obtained by substituting the PCEs of discrete variables into discrete equations, multiplying all members of such equations by polynomial $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ and taking the expected values. In this way exact stochastic Ampére's law and exact stochastic solenoidality equations of magnetic induction are obtained, together with approximate stochastic discrete constitutive relations. By combining these equations, an underdetermined system of equations follows

$$(1_n \otimes \mathbf{C}^T)\mathcal{M}_\nu(1_n \otimes \mathbf{C})\boldsymbol{A} = \tilde{\boldsymbol{J}}, \tag{3}$$

in which $\mathcal{M}_\nu$ is the symmetric, positive definite stochastic reluctivity matrix, while $\otimes$ indicates tensor product.

IV. Efficient Solution of Stochastic FIT equations

For non-intrusive stochastic FIT, the number of the required deterministic simulations can be reduced by using standard sparse grid techniques [5].

For intrustive stochastic FIT the conjugate gradient solver for deterministic discrete magnetostatic problems can still be used. However, in this way the computational complexity becomes unbearable for large n. This is due both to the storage requirement which is proportional to n^2 and to the number of operations at each iteration step which is also proportional to n^2. A huge reduction of computational complexity is achieved exploiting the fact that in usual magnetostatics problems, reluctivity is given by a random variable $\nu_k(\boldsymbol{\xi})$ in each subregion Ω_k made of a distinct material, with $k = 1, \ldots, r$. In this way, the discrete constitutive matrix takes the form

$$\mathcal{M}_\nu = \sum_{k=1}^{r} \mathbf{R}_\nu^k \otimes \mathbf{N}_\nu^k,$$

in which $\mathbf{R}_\nu^k = [r_{\nu,\boldsymbol{\alpha}\boldsymbol{\beta}}^k]$ are symmetric positive definite matrices of order n, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are multi-indexes, while $\mathbf{N}_\nu^k$ are constitutive matrices of deterministic problems. Then (3) can be rewritten by the set of n linear systems of equations

$$\sum_{k=1}^{r} \mathbf{C}^T \mathbf{N}_\nu^k \mathbf{C} \sum_{|\boldsymbol{\beta}|\leq p} r_{\nu,\boldsymbol{\alpha}\boldsymbol{\beta}}^k \boldsymbol{a}_{\boldsymbol{\beta}} = \tilde{\boldsymbol{j}}_{\boldsymbol{\alpha}} \tag{4}$$

for all n multi-indices $\boldsymbol{\alpha}$, with $|\boldsymbol{\alpha}|\leq p$.

Using this formulation, the storage requirement for the conjugate gradient algorithm is reduced to about that of *one* deterministic discrete magnetostatics problem. Also, the number of operations at each iteration step is proportional to n. Moreover, in *all* numerical tests it is observed that the number of iterations for reaching a specified residual is about the same for both deterministic and stochastic problems. In this way the total number of operations to reach convergence is proportional to n.

V. Numerical results

The magnetic circuit shown in Fig. 1 is analyzed. The magnetostatic problem is discretized by using a tetrahedral primal grid of about $50\,000$ tetrahedra and by its barycentric dual grid. For each of the four random variables 5-th order PCEs are assumed. In the non-intrusive approach using the pseudo-spectral method, each deterministic FIT equation (1) is solved for values of the random variables in a Smolyak sparse grid by the conjugate gradient method, until residual is reduced by 10^6 in about $2\,300$ iterations and in 1 hour 15 minutes of computational time. In the intrusive approach (4) is iteratively solved by the conjugate gradient method until residual is reduced by 10^6, by $2\,400$ iterations, in about 16 minutes. The probability density functions (pdf) of the flux φ of magnetic induction through section Σ are compared in Fig. 2 and exhibit a 0.1% agreement. Monte Carlo analysis of $10\,000$ tries and about 30 hours of computational time is in 5% agreement with these results.

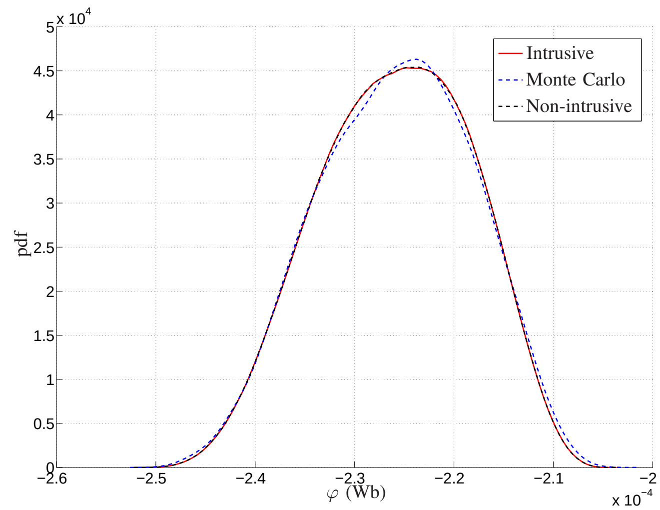

Fig. 2. Probability density function of the flux φ through Σ, estimated by both the intrusive, non-intrusive and MC methods.

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